

## BANACH SPACES WITH POLYNOMIAL NUMERICAL INDEX 1

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ABSTRACT. We characterize Banach spaces with polynomial numerical index 1 when they have the Radon-Nikodým property. The holomorphic numerical index is introduced and the characterization of the Banach space with holomorphic numerical index 1 is obtained when it has the Radon-Nikodým property.

## 1. INTRODUCTION

Let  $X$  be a Banach space over a real or complex scalar field  $\mathbb{K}$  and  $X^*$  its dual space. We denote by  $\mathcal{L}(X)$  the Banach space of all bounded linear operators from  $X$  to itself with usual operator norm. We consider the topological subspace  $\Pi(X) = \{(x, x^*) : x^*(x) = 1 = \|x\| = \|x^*\|\}$  of the product space  $B_X \times B_{X^*}$ , equipped with norm and weak-\* topology on  $B_X$ , the unit ball of  $X$  and  $B_{X^*}$  respectively. It is easy to see that  $\Pi(X)$  is a closed subspace of  $B_X \times B_{X^*}$ . The **numerical radius**  $v(T)$  of a linear operator  $T : X \rightarrow X$  is defined by  $v(T) = \sup\{|x^*Tx| : (x, x^*) \in \Pi(X)\}$ . We denote by  $n(X)$  the **numerical index** of  $X$  defined by  $n(X) = \inf\{v(T) : T \in \mathcal{L}(X), \|T\| = 1\}$ .

Notice that  $0 \leq n(X) \leq 1$  and  $n(X)\|T\| \leq v(T) \leq \|T\|$  for any  $T \in \mathcal{L}(X)$ . Hence  $v(\cdot)$  is equivalent to the operator norm on  $\mathcal{L}(X)$  when  $n(X) > 0$ . For more properties about the numerical radius and index, see [10]. As in [9], the notion of numerical radius can be extended to elements in  $C_b(B_X : X)$  of bounded continuous functions from  $B_X$  to  $X$ . More precisely, for each  $f \in C_b(B_X : X)$ ,  $v(f) = \sup\{|x^*f(x)| : (x, x^*) \in \Pi(X)\}$ . Notice that  $C_b(B_X : Y)$  is a Banach space equipped with sup norm  $\|f\| = \sup\{\|f(x)\| : x \in B_X\}$ .

Let  $X$  and  $Y$  be Banach spaces and  $k \geq 1$ . A bounded  $k$ -homogeneous polynomial  $P : X \rightarrow Y$  is defined to be  $P(x) = L(x, \dots, x)$  for all  $x \in X$ , where  $L : X \times \dots \times X \rightarrow Y$  is a continuous  $k$ -multilinear map. We denote by  $\mathcal{P}({}^kX : Y)$  the Banach space of all bounded  $k$ -homogeneous polynomials from  $X$  to  $Y$  as a subspace of  $C_b(B_X : Y)$ . Following the notations in [2], the  **$k$ -polynomial numerical index** of a Banach space  $X$  is defined to be  $n^{(k)}(X) = \inf\{v(P) : P \in \mathcal{P}({}^kX : X), \|P\| = 1\}$ . It is easy to see that  $n^{(k+1)}(X) \leq n^{(k)}(X)$  for any  $k \geq 1$ .

Let  $X$  be a real or complex Banach space and let  $K$  be a nonempty convex subset of  $X$ . Recall that  $x \in K$  is said to be an **extreme point** of  $K$  if whenever  $y + z = 2x$  for some  $y, z \in K$ , we have  $x = y = z$ . Denote by  $\text{ext}(K)$  the set of all extreme points of  $K$ .

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1991 *Mathematics Subject Classification.* 46G25 (primary), 46B20, 46B22 (secondary).

This work was supported by the Korea Research Foundation Grant funded by the Korean Government(MOEHRD) (KRF-2006-352-C00003) and by grant No. R01-2004-000-10055-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

The geometric properties of a Banach space  $X$  with  $n(X) = 1$  have been studied [11, 12, 13]. McGregor [13] gave a geometric characterization of finite dimensional Banach spaces with numerical index 1. More precisely, a finite dimensional Banach space  $X$  has numerical index 1 if and only if  $|x^*(x)| = 1$  for every  $x \in \text{ext}(B_X)$  and for every  $x^* \in \text{ext}(B_{X^*})$ . For a Banach space  $X$  with the Radon-Nikodým property, it was shown [11, 12] (cf. [10, Corollary 1]) that  $n(X) = 1$  if and only if  $|x^*(x)| = 1$  for every  $x^* \in \text{ext}(B_{X^*})$  and for every denting point  $x$  of  $B_X$ . For the definition of denting point, see [5]. In [10], they asked if there are similar characterizations of the Banach space  $X$  with  $n^{(k)}(X) = 1$  for each  $k \geq 2$ . In this paper, we give the partial answer.

We need the following notion which is introduced by Ferrera [7] for the  $k$ -homogeneous polynomials.

**Definition 1.1.** Let  $X$  and  $Y$  be Banach spaces over  $\mathbb{K}$ . A nontrivial function  $f \in C_b(B_X : Y)$  is said to **strongly attain its norm** at  $x_0$  if whenever  $\lim_n \|f(x_n)\| = \|f\|$  for a sequence  $\{x_n\}$  in  $B_X$ , it has a subsequence  $\{x_{n_k}\}_{k=1}^\infty$  converging to  $\alpha x_0$  for some  $|\alpha| = 1$ ,  $\alpha \in \mathbb{K}$ . Let  $H$  be a subspace of  $C_b(B_X : Y)$ . Denote by  $\tilde{\rho}H$  the set

$$\tilde{\rho}H = \{x_0 : f \text{ strongly attains its norm at } x_0 \text{ and } f \in H\}.$$

A nonzero function  $f \in C_b(B_X : Y)$  is said to be a **strong peak function** at  $x_0$  if whenever there is a sequence  $\{x_n\}_{n=1}^\infty$  in  $B_X$  with  $\lim_n \|f(x_n)\| = \|f\|$ , the sequence  $\{x_n\}_n$  converges to  $x_0$ . We denote by  $\rho H$  the set

$$\rho H = \{x_0 : f \text{ is a strong peak function in } H \text{ at } x_0\}.$$

Notice that  $f \in C_b(B_X : Y)$  strongly attains its norm at  $x$  if and only if for each  $\epsilon > 0$ , there is  $\delta > 0$  such that whenever  $\|f(y)\| \geq \|f\| - \delta$  for some  $y \in B_X$ , we get  $\inf_{\lambda \in S_{\mathbb{K}}} \|\lambda x - y\| \leq \epsilon$ . Notice also that  $\tilde{\rho}\mathcal{P}({}^k X : Y) = \tilde{\rho}\mathcal{P}({}^k X)$  for each  $k \geq 1$  if  $Y$  is nontrivial.

For two complex Banach spaces  $X$  and  $Y$ , we are interested in two subspaces of  $C_b(B_X : Y)$ ,

$$\begin{aligned} A_b(B_X : Y) &= \{f \in C_b(B_X : Y) : f \text{ is holomorphic on the open unit ball } B_X^\circ\} \\ A_u(B_X : Y) &= \{f \in A_b(B_X : Y) : f \text{ is uniformly continuous on } B_X\}. \end{aligned}$$

We denote by  $A(B_X : Y)$  either  $A_u(B_X : Y)$  or  $A_b(B_X : Y)$ . When  $Y = \mathbb{C}$ , we write  $A(B_X)$  instead of  $A(B_X : \mathbb{C})$ . By the maximum modulus theorem, it is easy to see that if  $f \in A_b(B_X : Y)$  strongly attains its norm at  $x_0$  and  $X$  is nontrivial, then  $x_0$  is contained in  $S_X$ , the unit sphere of  $X$ .

## 2. MAIN RESULTS

Let  $X$  and  $Y$  be complex Banach spaces. It is shown (see the proof of [4, Theorem 4.4]) that if  $f \in A(B_X : Y)$  strongly attains its norm at  $x_0$ . Then, given  $\epsilon > 0$ , there is  $g \in A(B_X : Y)$  with  $\|g\| < \epsilon$  and  $f + g$  is a strong peak function at  $\alpha x_0$  for some  $\alpha \in S_{\mathbb{C}}$ . This implies that  $x_0$  is a strong peak point of  $A(B_X : Y)$ . So it is easy to see that  $\rho A(B_X) = \rho A(B_X : Y) = \tilde{\rho} A(B_X : Y) = \tilde{\rho} A(B_X)$ . Recall that  $x \in B_X$  is said to be a **complex extreme point** of  $B_X$  if  $\sup_{0 \leq \theta \leq 2\pi} \|x + e^{i\theta} y\| \leq 1$  for some  $y \in X$  implies  $y = 0$ . We denote by  $\text{ext}_{\mathbb{C}}(B_X)$  the set of all complex extreme points of  $B_X$ . Notice that every strong peak point of  $A(B_X)$  is a complex extreme point of  $B_X$  (see [8]). So we have the following.

**Proposition 2.1.** Let  $X$  and  $Y$  be complex Banach spaces and  $H$  a subspace of  $A(B_X : Y)$ . Then

$$\tilde{\rho}H \subset \rho A(B_X : Y) \subset \text{ext}_{\mathbb{C}}(B_X).$$

It is worth while to remark that when  $X$  is a nontrivial finite dimensional complex Banach space, then  $\rho A(B_X) = \text{ext}_{\mathbb{C}}(B_X)$  (see [3]).

Let  $X$  be a real or complex Banach space and let  $K$  be a nonempty convex subset of  $X$ . Recall that an element  $x$  in  $K$  is said to be a **strongly exposed point** of  $K$  if there is nonzero  $x^* \in B_{X^*}$  such that  $\text{Re } x^*(x) = \sup\{\text{Re } x^*(y) : y \in K\}$  and whenever  $\lim_n \text{Re } x^*(x_n) = \text{Re } x^*(x)$  for some sequence  $\{x_n\}_{n=1}^\infty$  in  $K$ , we get  $\lim_n \|x_n - x\| = 0$ . Notice that  $\tilde{\rho}\mathcal{P}({}^1X)$  is  $\text{sexp}(B_X)$ , the set of all strongly exposed points of  $B_X$ . Hence we get the following corollary.

**Corollary 2.2.** Let  $X$  be a real or complex Banach space. Then

$$\text{sexp}(B_X) = \tilde{\rho}\mathcal{P}({}^1X) \subset \tilde{\rho}\mathcal{P}({}^2X) \subset \dots$$

In particular, when  $X$  is a complex space,

$$\bigcup_{k=1}^{\infty} \tilde{\rho}\mathcal{P}({}^kX) \subset \rho A_u(B_X) \subset \rho A_b(B_X) \subset \text{ext}_{\mathbb{C}}(B_X).$$

*Proof.* We may assume that  $X \neq 0$  and need show that  $\tilde{\rho}\mathcal{P}({}^kX) \subset \tilde{\rho}\mathcal{P}({}^{k+1}X)$  for each  $k \geq 1$ . Suppose that  $y \in \tilde{\rho}\mathcal{P}({}^kX)$ . So  $y \in S_X$  and there is  $P \in \mathcal{P}({}^kX)$  which strongly attains its norm at  $y$ . Choose  $y^* \in X^*$  such that  $\|y^*\| = y^*(y) = 1$ . Define  $Q : X \rightarrow \mathbb{K}$  by  $Q(x) = y^*(x)P(x)$  for each  $x \in X$ . So  $Q$  is a  $(k+1)$ -homogeneous polynomial and it is easy to see that  $Q$  strongly attains its norm at  $y$ . The proof is complete.  $\square$

Recall that a Banach space  $X$  is said to have the **Radon-Nikodým property** if every nonempty bounded closed convex subset in  $X$  is a closed convex hull of its strongly exposed points [5]. Reviewing the proof of Theorem 4.2 in [4] (cf. [7]), we get the following.

**Proposition 2.3.** Let  $X, Y$  be Banach spaces over  $\mathbb{K}$ . Suppose that  $X$  has the Radon-Nikodým property and  $Y$  is nontrivial. Then for each  $P \in \mathcal{P}({}^kX : Y)$  and  $\epsilon > 0$ , there is  $Q \in \mathcal{P}({}^kX : Y)$  such that  $\|Q\| \leq \epsilon$  and  $P + Q$  strongly attains its norm.

Now we get the polynomial version of Bishop's theorem [1, 4]. Recall that a subset  $A$  of  $B_X$  is said to be **balanced** if  $\lambda \in S_{\mathbb{K}}$  implies  $\lambda A \subset A$ .

**Proposition 2.4.** Let  $k$  be a positive integer and  $X, Y$  Banach spaces over  $\mathbb{K}$ . Suppose that  $X$  has the Radon-Nikodým property and  $Y$  is nontrivial. Then the set  $\tilde{\rho}\mathcal{P}({}^kX)$  is a norming subset of  $\mathcal{P}({}^kX : Y)$ . In fact, the closure of  $\tilde{\rho}\mathcal{P}({}^kX)$  is the smallest closed balanced norming subset of  $\mathcal{P}({}^kX : Y)$ .

*Proof.* By Proposition 2.3, for each  $P \in \mathcal{P}({}^kX : Y)$ , there is a sequence  $\{P_n\}_{n=1}^\infty$  in  $\mathcal{P}({}^kX : Y)$  such that  $\lim_n \|P_n - P\| = 0$  and each  $P_n$  strongly attains its norm at  $x_n$ . So  $\|P_n\| = \|P_n(x_n)\|$ . Then for each  $n \geq 1$ ,

$$\|P_n(x_n)\| - \|P_n - P\| \leq \|P(x_n)\| \leq \|P_n(x_n)\| + \|P_n - P\|$$

holds. So  $\lim_n \|P(x_n)\| = \lim_n \|P_n(x_n)\| = \lim_n \|P_n\| = \|P\|$ . This shows that  $\tilde{\rho}\mathcal{P}({}^kX)$  is a norming subset of  $\mathcal{P}({}^kX : Y)$ . So it is clear that the closure of

$\tilde{\rho}\mathcal{P}({}^kX)$  is a closed norming balanced subset of  $S_X$ . Suppose that  $A$  is a closed balanced norming subset of  $\mathcal{P}({}^kX : Y)$ . Let  $P$  be a strongly norm-attaining element at  $x_0$ . Since  $A$  is norming, choose a sequence  $\{x_n\}_{n=1}^\infty$  such that  $\|P\| = \lim_n \|P(x_n)\|$ . Then we get a subsequence  $\{y_l\}_{l=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$  and  $\lambda \in S_{\mathbb{K}}$  such that  $\lim_l y_l = \lambda x_0$ . So  $x_0$  is contained in  $A$ . This shows that the closure of  $\tilde{\rho}\mathcal{P}({}^kX)$  is contained in  $A$ . This completes the proof.  $\square$

The following Proposition 2.5 and Theorem 2.6 are related with Problem 45 in [10]. Similar characterizations were shown in [11, 12, 13] for Banach spaces with the numerical index 1. Recall that a point  $x^* \in B_{X^*}$  is called a **weak-\* exposed point** of  $B_{X^*}$  if there is  $x \in B_X$  such that  $x^*(x) = 1$  and  $y^*(x) = 1$  for some  $y^* \in B_{X^*}$  implies  $y^* = x^*$ . The corresponding point  $x$  is said to be a **smooth point** of  $S_X$ . We denote by  $w^*\exp(B_{X^*})$  the set of all weak-\* exposed points of  $B_{X^*}$ . In the proof of Theorem 2.6, we use the Mazur theorem which says that if a Banach space  $X$  is separable, then the set of all smooth points of  $S_X$  is dense in  $S_X$ .

**Proposition 2.5.** Let  $k \geq 1$  be a positive integer and  $X$  a real or complex Banach space with  $n^{(k)}(X) = 1$ . Then  $|x^*(x)| = 1$  for each  $x \in \tilde{\rho}\mathcal{P}({}^kX)$  and  $x^* \in \text{ext}(B_{X^*})$ .

*Proof.* Suppose that  $n^{(k)}(X) = 1$ . Fix  $x_0^* \in \text{ext}(B_{X^*})$  and take  $P$  which strongly attains its norm at  $x_0$  with  $\|P\| = 1$ . We need show that  $|x_0^*(x_0)| = 1$ .

Fix  $\epsilon > 0$ . Because  $P$  strongly attains its norm at  $x_0$ , there is  $\alpha > 0$  such that  $|P(y)| \geq 1 - \alpha$  for some  $y \in B_X$  implies  $\inf_{\lambda \in S_{\mathbb{K}}} \|\lambda x - y\| \leq \epsilon$ .

Set  $F = \{x^* \in B_{X^*} : |(x^* - x_0^*)(x_0)| \geq \epsilon\}$ . Then  $F$  is weak-\* compact and  $x_0^*$  is not contained in  $\overline{\text{co}}^*(F)$ , the weak-\* closure of convex hull of  $F$ . Otherwise,  $x_0^*$  is an extreme point of  $\overline{\text{co}}^*(F)$  and  $x_0^* \in F$  by the converse Krein-Milman theorem. Hence there exist  $y \in S_X$  and  $\beta > 0$  such that

$$\text{Re } x_0^*(y) > 1 - \beta \geq \text{Re } x^*(y), \quad \forall x^* \in F.$$

That is, if  $\text{Re } x^*(y) > 1 - \beta$  for some  $x^* \in B_{X^*}$ , then  $|(x^* - x_0^*)(x_0)| < \epsilon$ .

Let  $f(x) = P(x)y$  for each  $x \in X$ . Then  $f \in \mathcal{P}({}^kX : X)$  and  $v(f) = \|f\| = 1$ . Take  $\delta = \min\{\alpha, \beta\}$ . Then there is  $(x, x^*) \in \Pi(X)$  such that  $|x^*f(x)| > 1 - \delta$ . This means that  $|P(x)| > 1 - \alpha$  and  $|x^*(y)| = \text{Re } (\lambda_1 x^*)(y) > 1 - \beta$  for some  $\lambda_1 \in S_{\mathbb{K}}$ . Hence  $\inf_{\lambda \in S_{\mathbb{K}}} \|\lambda x_0 - x\| = \|\lambda_2 x_0 - x\| \leq \epsilon$  for some  $\lambda_2 \in S_{\mathbb{K}}$  and  $|(\lambda_1 x^* - x_0^*)(x_0)| < \epsilon$ . By the triangle inequality,

$$\begin{aligned} ||x^*(x_0)| - |x^*(x)|| &\leq |x^*(\lambda_2 x_0 - x)| \leq \|\lambda_2 x_0 - x\| \leq \epsilon \quad \text{and} \\ |x^*(x_0)| - |x_0^*(x_0)| &\leq |(\lambda_1 x^* - x_0^*)(x_0)| \leq \epsilon. \end{aligned}$$

Notice that  $x^*(x) = 1$ . So  $|1 - |x_0^*(x_0)|| \leq 2\epsilon$ .

Because  $\epsilon > 0$  is arbitrary,  $|x_0^*(x_0)| = 1$ . This completes the proof.  $\square$

**Theorem 2.6.** Suppose that  $k$  is a positive integer and  $X$  is a nontrivial real or complex Banach space with the Radon-Nikodým property. The following are equivalent.

- (1)  $n^{(k)}(X) = 1$ .
- (2)  $|x^*(x)| = 1$  for each  $x \in \tilde{\rho}\mathcal{P}({}^kX)$  and  $x^* \in \text{ext}(B_{X^*})$ .

In addition, when  $X$  is separable,  $n^{(k)}(X) = 1$  if and only if  $|x^*(x)| = 1$  for each  $x \in \tilde{\rho}\mathcal{P}({}^kX)$  and  $x^* \in w^*\exp(B_{X^*})$ .

*Proof.*  $1 \Rightarrow 2$  is proved by Proposition 2.5. Conversely, suppose that we have 2. By Proposition 2.3, we have only to show that  $v(f) = \|f\|$  for every strongly norm-attaining element  $f \in \mathcal{P}({}^k X : X)$ . Fix a strongly norm-attaining element  $f$  in  $\mathcal{P}({}^k X : X)$  with  $\|f\| = \|f(x)\|$ . Then the set  $F = \{y^* \in B_{X^*} : y^* f(x) = \|f(x)\|\}$  is a nonempty weak-\* compact convex subset of  $B_{X^*}$ . By the Krein-Milman theorem, there is an extreme point  $x^*$  of  $F$  and it is easy to see that  $x^*$  is also an extreme point of  $B_{X^*}$ . Hence  $|x^*(x)| = 1$  by the assumption and  $v(f) \geq |x^* f(x)| = \|f(x)\| = \|f\|$ . This proves  $2 \Rightarrow 1$ .

Suppose that  $X$  is separable. Because  $w^* \exp(B_{X^*}) \subset \text{ext}(B_{X^*})$ , the “only if” part is clear by Proposition 2.5. So we need prove the sufficiency. By Proposition 2.3, it is enough to show that  $v(f) = \|f\|$  for every strongly norm-attaining element  $f \in \mathcal{P}({}^k X : X)$ . Fix a strongly norm-attaining element  $f$  in  $\mathcal{P}({}^k X : X)$  with  $\|f\| = \|f(x)\|$ . Since  $X$  is separable, the set of smooth points in  $S_X$  is dense. Given  $\epsilon > 0$ , choose a smooth point  $y \in S_X$  with

$$\left\| y - \frac{f(x)}{\|f\|} \right\| \leq \frac{\epsilon}{\|f\|}.$$

Then there is  $x^* \in w^* \exp(B_{X^*})$  such that  $x^*(y) = 1$  and  $|x^* f(x)| \geq \|f\| - \epsilon$ . So  $|x^*(x)| = 1$  by assumption. Hence  $v(f) \geq |x^* f(x)| \geq \|f\| - \epsilon$ . Since  $\epsilon > 0$  is arbitrary, we get  $v(f) = \|f\|$ . This completes the proof.  $\square$

Now we shall give a partial answer to Problem 44 in [10], which ask if the only real Banach space  $X$  with  $n^{(2)}(X) = 1$  is  $X = \mathbb{R}$ .

**Theorem 2.7.** Suppose that  $X$  is a real nontrivial finite dimensional Banach space. Then  $\tilde{\rho}\mathcal{P}({}^2 X) = S_X$ . In addition, if  $n^{(2)}(X) = 1$  then  $X = \mathbb{R}$ .

*Proof.* Suppose that  $\dim X = n$  and identify  $X$  with  $X^{**}$ . The first assertion is clear when  $n = 1$ . So we may assume that  $n \geq 2$ . Fix  $x_1 \in S_X$  and choose  $x_1^* \in S_{X^*}$  with  $x_1^*(x_1) = 1$ . Then there is a linearly independent subset  $\{x_k^*\}_{k=2}^n$  in  $B_{\ker x_1}$ . So  $\{x_i^*\}_{i=1}^n$  is a basis of  $X^*$ .

Fix a positive sequence  $\{\epsilon_k\}_{k=1}^n$  with  $\sum_{k=1}^n \epsilon_k < 1/2$ . Then define a 2-homogeneous polynomial

$$P(x) = x_1^*(x)^2 - \sum_{k=2}^n \epsilon_k x_k^*(x)^2.$$

Notice that  $\|P\| = \|P(x_1)\| = 1$ . We shall show that  $P$  strongly attains its norm at  $x_1$ . Suppose that there is a sequence  $\{y_m\}_{m=1}^\infty$  in  $B_X$  with  $\lim_m |P(y_m)| = 1$ . This shows that  $\lim_m |x_1^*(y_m)| = 1$  and  $\lim_m |x_k^*(y_m)| = 0$  for every  $k \geq 2$ . Since  $B_X$  is compact, we can choose a proper subsequence of  $\{y_m\}_m$  which converges to  $\pm x_1$ . This shows that  $x_1$  is in  $\tilde{\rho}\mathcal{P}({}^2 X)$ . This proves the first assertion.

In addition, if  $n^{(2)}(X) = 1$ , then  $|x^*(x)| = 1$  for every  $x \in S_X = \tilde{\rho}\mathcal{P}({}^2 X)$  and  $x^* \in \text{ext}(B_{X^*})$  by Proposition 2.5. By the Minkowski theorem, there is  $x^* \in \text{ext}(B_{X^*})$ . Then  $|x^*(x)| = 1$  for every  $x \in S_X$ . If  $\dim X \geq 2$ , then there is  $x \in S_X$  such that  $x^*(x) = 0$ . This is a contradiction.  $\square$

Now we denote by  $n_u(X)$  the **holomorphic numerical index** defined by  $n_u(X) = \inf\{v(f) : f \in A_u(B_X : X), \|f\| = 1\}$ . The proof of Proposition 2.5 gives the following.

**Proposition 2.8.** Let  $X$  be a complex Banach space with  $n_u(X) = 1$ . Then  $|x^*(x)| = 1$  for each  $x \in \rho A_u(B_X)$  and  $x^* \in \text{ext}(B_{X^*})$ .

The proof of Theorem 2.6 gives the following if we use Proposition 2.8 instead of Proposition 2.5 and the fact that the set of all strong peak functions in  $A_u(B_X : X)$  is dense [4].

**Proposition 2.9.** Suppose that  $X$  is a nontrivial complex Banach space with the Radon-Nikodým property. The following are equivalent.

- (1)  $n_u(X) = 1$ .
- (2) For each  $x \in \rho A_u(B_X)$  and  $x^* \in \text{ext}(B_{X^*})$ , we have  $|x^*(x)| = 1$ .

In addition, when  $X$  is separable,  $n_u(X) = 1$  if and only if  $|x^*(x)| = 1$  for each  $x \in \rho A_u(B_X)$  and  $x^* \in w^*\text{exp}(B_{X^*})$ .

**Corollary 2.10.** Suppose that  $X$  is a finite dimensional complex Banach space. The following are equivalent.

- (1)  $n_u(X) = 1$ .
- (2) For each  $x \in \text{ext}_{\mathbb{C}}(B_X)$  and  $x^* \in \text{ext}(B_{X^*})$ , we have  $|x^*(x)| = 1$ .

*Proof.* Notice that if  $X$  is finite dimensional,  $\rho A_u(B_X) = \text{ext}_{\mathbb{C}}(B_X)$  (see [3, Proposition 1.1]). So  $1 \iff 2$  is shown by Proposition 2.9. This completes the proof.  $\square$

**Acknowledgements.** The author thanks Yun Sung Choi. He introduced this topic to the author and kindly shared his ideas which improved the previous version of this paper.

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